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Relativistic Physics from Quantum Theory

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An investigation of the invariance of quantum theory under the complex group reveals a natural origin of relativistic physics from quantum theory. Once such an origin of relativity is accepted, quantum limitations on the applicability of standard relativistic theory also become evident.

1. INTRODUCTION

It has become an accepted part of modern physical theory that the origin of the electromagnetic interaction is a simple consequence of the invariance of pure quantum states under the action of the group $U(1) = \{e^{i\theta}: \theta \in R\}$, where i is the complex structure of the Hilbert space $\mathscr{H}$ whose rays mathematically represent the pure states [Hermann Weyl's gauge theory (Weyl, 1931)]. However, the rays of complex projective space, $\mathscr{P}_{\mathscr{H}}\mathscr{C}$, are invariant not only under the compact Abelian group $U(1)$, but also under the action of the noncompact Abelian group $\{e^{\rho}: \rho \in R\}$, i.e., we have invariance under the full complex group $\mathscr{C}_0 = \mathscr{C} - \{0\} = \{e^{\rho+i\theta}: \rho, \theta \in R\}$, where $\mathscr{C}$ represents the action of the complex field on $\mathscr{H}$. This group is a two-dimensional real Lie group, but to the present author's knowledge only one dimension has been given physical interpretation in quantum theory and no one seems to have investigated the question of whether the noncompact component has any analogous interpretation. This paper is an investigation of that question. We start with a formulation of the differential geometry of an arbitrary n-dimensional complex projective space, but quickly specialize to the case of projective complex 1-space $[\mathscr{P}_1\mathscr{C}]$, simply because we uncover the interesting fact that, until the strong interaction (or beyond) predominates, all of modern physics can be covered by that simplest case. In the process we also formulate a generalization of standard quantum theory that allows all non-field-theoretical computations to be

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carried out in finite dimensions rather than the usual infinite-dimensional Hilbert space context, so that we need only finite-dimensional projective spaces as base spaces for all constructions. Field theory then naturally follows as the theory of associated bundle sections over the base space. Remarkably, we find that a full general relativistic theory is intrinsic to the geometry of $\mathscr{P}_1\mathscr{C}$ quantum systems, and this relativistic structure is naturally identifiable with the corresponding structure of the real world to the extent that our observations of the universe are basically determined by $\mathscr{P}_1\mathscr{C}$ structures, e.g., spin-1/2 electronic structure of atomic matter.

2. $\mathscr{P}_n\mathscr{C}$ AND ITS DIFFERENTIABLE STRUCTURE

To construct $\mathscr{P}_n\mathscr{C}$ and specify clearly its differential geometry, we start with even-dimensional real space $R^{2(n+1)}$ equipped with its standard Euclidean topological and differentiable structures, and choose some complex structure operator $i \in \mathscr{R} \equiv R^{2(n+1)} \otimes R^{2(n+1)}$ such that $i^2 = -I \in \mathscr{R}$ and $i = -i^+$, where of course i^+ means the adjoint operator relative to the usual Euclidean metric on $R^{2(n+1)}$. Let $\mathscr{R}^i \subset \mathscr{R}$ be the commutant of i, and let $\mathscr{C} = \{aI + bi\colon\ a, b \in R\} \subset \mathscr{R}^i$ be the resulting complex field acting on $R^{2(n+1)}$, with $\mathscr{C}_0 = \mathscr{C} - \{0\} = R_+ U(1) = \{e^{\rho I + \theta i}\colon\ \rho, \theta \in R\}$ the complex group, decomposable into the compact unitary and noncompact positive subgroups $U(1)$ and R_+, respectively. Finally, let $R_0^{2(n+1)} = R^{2(n+1)} - \{0\}$. Then

$$\mathscr{P}_n\mathscr{C} = R_0^{2(n+1)}/\mathscr{C}_0 = \left\{\hat{x} = \{cx\colon\ c \in \mathscr{C}_0\}\colon\ x \in R_0^{2(n+1)}\right\}$$

i.e., $\mathscr{P}_n\mathscr{C}$ is the space of orbits $\hat{x} = \mathscr{C}_0 x \subset R_0^{2(n+1)}$ of the group $\mathscr{C}_0$ in $R_0^{2(n+1)}$.

To construct now the standard differentiable structure on $\mathscr{P}_n\mathscr{C}$ we need to introduce an atlas of charts, and this can be most conveniently done in physically interpretable terms using the structure of the $\mathscr{C}$-linear operator space $\mathscr{R}^i$. We note that each $\hat{x} \in \mathscr{P}_n\mathscr{C}$ uniquely defines a two-dimensional subspace $\hat{x} \cup \{0\} = \mathscr{C}x = \{(aI + bi)x\} \subset R^{2(n+1)}$, where x can be any element of the orbit $\hat{x}$, and so there is a well-defined correspondence between elements $\hat{x} \in \mathscr{P}_n\mathscr{C}$ and subsets $\Psi_{\hat{x}} \subset \mathscr{R}^i$ of $\mathscr{C}$-linear idempotent operators that map $R^{2(n+1)}$ onto $\hat{x} \cup \{0\}$ (two-dimensional projections onto $\hat{x} \cup \{0\}$ that commute with i). Writing elements of $\Psi_{\hat{x}}$ as $\hat{\psi}_{\hat{x}}, \hat{\phi}_{\hat{x}}, \ldots$, and defining the complementary projections $\hat{\psi}_{\hat{x}}^{\perp} = I - \hat{\psi}_{\hat{x}} \in \Psi_{\hat{x}}^{\perp} = I - \Psi_{\hat{x}}$, we obviously have

$$\hat{\phi}_{\hat{x}}\hat{\psi}_{\hat{x}} = \hat{\psi}_{\hat{x}}$$

$$\hat{\phi}_{\hat{x}}^{\perp}\psi_{\hat{x}} = 0$$

and the implications

$$\hat{\varphi}_{\hat{x}}\hat{\psi}_{\hat{x}} = \hat{\psi}_{\hat{x}}\hat{\varphi}_{\hat{x}} \quad \text{iff} \quad \hat{\varphi}_{\hat{x}} = \hat{\psi}_{\hat{x}}$$

$$\hat{\varphi}_{\hat{x}}\hat{\psi}_{\hat{x}}^{\perp} = 0 \quad \text{iff} \quad \hat{\varphi}_{\hat{x}} = \hat{\psi}_{\hat{x}}$$

Adopting physically motivated terminology, we refer to the elements $\hat{x} \in \mathscr{P}_n\mathscr{C}$ as *states*, and each $\hat{\psi}_{\hat{x}} \in \Psi_{\hat{x}}$ will be called a *projective representation* or *realization* of the state $\hat{x}$ (also commonly called a *density operator representation*).

So far, then, we have a class of trace-2 projections corresponding to each element $\hat{x}$ of projective space. However, picking out a metric g on $R^{2(n+1)}$ gives us a meaning for adjoint operators in $\mathscr{R}$ in the standard way, and hence provides also a meaning for self-adjointness in $\mathscr{R}$. This allows us to pick out from each set $\Psi_{\hat{x}}$ the unique element $\psi_{\hat{x}}$ that is self-adjoint in the given metric, and this in turn gives us embeddings $\hat{\psi}$ of $\mathscr{P}_n\mathscr{C}$ into the self-adjoint operators in $\mathscr{R}^i$ relative to the chosen metric. When the standard metric g_0 on $R^{2(n+1)}$ is used for this construction we will refer to the resulting embedding $\hat{\psi}$ as the *standard self-adjoint projective representation* of $\mathscr{P}_n\mathscr{C}$ in the operator space $\mathscr{R}^i$. This representation is the one we will use for the construction of an atlas of charts on $\mathscr{P}_n\mathscr{C}$, but for later purposes it must be emphasized that any equivalent metric could be used, so that our resulting differentiable structure is independent of such choices of equivalent metric. Now a choice of metric (say, g_0) also gives us a meaning for orthogonality in $R^{2(n+1)}$, and, using adjointness, an induced metric trace (A^+B) on the operator space $\mathscr{R}$, with the corresponding meaning for orthogonality in $\mathscr{R}$ and $\mathscr{R}^i$. This allows us to choose exactly $n+1$ mutually orthogonal orbits $\hat{x}_j \in \mathscr{P}_n\mathscr{C}$ (defining orthogonal planes in $R^{2(n+1)}$) with corresponding mutually orthogonal self-adjoint projection operators $\hat{\psi}_j = \hat{\psi}_{\hat{x}_j} \in \mathscr{R}^i$.

Using the standard metric to make all these choices, we can define $n+1$ subsets of $\mathscr{P}_n\mathscr{C}$:

$$\mathscr{P}_n^j\mathscr{C} = \{\hat{x} \in \mathscr{P}_n\mathscr{C}\colon \hat{\psi}_j\hat{\psi}_x \neq 0\}$$

to be used as chart domains. As $\hat{x}$ ranges over the jth domain $\mathscr{P}_n^j\mathscr{C}$ it is straightforward to check that the mapping

$$X_j\colon \mathscr{P}_n^j\mathscr{C} \to \mathscr{R}^i$$

$$\hat{x} \to X_j(\hat{x}) = \hat{\psi}_j^{\perp}\hat{\psi}_{\hat{x}}\hat{\psi}_j + \hat{\psi}_j\hat{\psi}_{\hat{x}}\hat{\psi}_j^{\perp}$$

is bijective onto an open subset of the $2n$-dimensional subspace $\mathscr{R}^i_{\hat{\psi}_j} \subset \mathscr{R}^i$ consisting of those self-adjoint trace zero $\mathscr{C}$-linear operators that map the orbit $\hat{x}_j \subset R^{2(n+1)}$ into the complementary range space of $\hat{\psi}_j^{\perp}$ in $R^{2(n+1)}$, and that $X_j \circ X_k^{-1}$ (well defined on an open set in $X_k(\mathscr{P}_n^k\mathscr{C}) \subset \mathscr{R}^i_{\hat{\psi}_k}$) is a diffeomorphism in terms of the standard differentiable structure of operators on $R^{2(n+1)}$. Taking the $n+1$ mappings X_j as charts, we have the standard differentiable structure on $\mathscr{P}_n\mathscr{C}$ as a $2n$-dimensional real manifold or, of course, an n-complex-dimensional manifold. The standard Riemannian metric g_0 on $\mathscr{P}_n\mathscr{C}$ (Fubini–Study metric) is simply the obvious metric induced by the trace metric on the operator spaces $\mathscr{R}^i_{\hat{\psi}_j} \subset \mathscr{R}^i$ (considered of course as tangent vector representation spaces for $\mathscr{P}_n\mathscr{C}$). Similarly, the standard symplectic form is defined by the metric $\mathrm{tr}(A^+B)$ applied to the operators iA, B, i.e.,

$$\omega_0(A, B) = \mathrm{tr}([iA]^+ B) = -\mathrm{tr}(iA^+B)$$

Together the two forms define the standard Hermitian metric

$$\begin{aligned} h_0(A, B) &= g_0(A, B) + i\omega_0(A, B) \\ &= \mathrm{tr}(A^+B) - i\,\mathrm{tr}(iA^+B) \end{aligned}$$

with the required properties

$$h_0(CA, B) = \bar{C}h_0(A, B)$$

$$h_0(A, CB) = Ch_0(A, B)$$

We emphasize again that if we had chosen a different but equivalent metric on $R^{2(n+1)}$ we would arrive at the same differentiable structure for $\mathscr{P}_n\mathscr{C}$, but with different (but equivalent) forms g, ω, h over $\mathscr{P}_n\mathscr{C}$.

Finally, while, as we will see in the next section, the embedding $\hat{\psi}$ is given immediate physical interpretation by quantum theory, the most directly laboratory-measurable quantities seem to be more simply expressed in terms of an auxiliary construction, only trivially different from $\hat{\psi}$—the *polarization vector or spin vector* σ corresponding to $\hat{\psi}$, defined for each state $\hat{x} \in \mathscr{P}_n\mathscr{C}$ as the trace 0, unit length self-adjoint operator

$$\sigma_{\hat{x}} = N_n\left(\hat{\psi}_{\hat{x}} - n^{-1}\hat{\psi}^{\perp}_{\hat{x}}\right) = N_n\left(\left[1 + n^{-1}\right]\hat{\psi}_{\hat{x}} - n^{-1}I\right)$$

where the normalization factor is $N_n = n^{1/2}[2(n+1)]^{-1/2}$. [See Schiff, (1968, pp. 382ff. for physical discussion.] Now trace 0 simply means orthogonality with respect to the operator $I \in \mathscr{R}^i$, and this condition obviously defines a

subspace $\mathscr{R}^i_{S0} \subset \mathscr{R}^i$ [the space of $\mathscr{C}$-linear trace-0 self-adjoint operators]. Inspection of our chart formula above shows that, for $\hat{x} \in \mathscr{P}^j_n\mathscr{C}$,

$$X_j(\hat{x}) = P_j \cdot \hat{\psi}_{\hat{x}} = 2N_n P_j \cdot \sigma_{\hat{x}}$$

where P_j is the projection operator defined *on* operator space $\mathscr{R}$ by

$$P_j = \hat{\psi}_j^{\perp}[\ ]\hat{\psi}_j + \hat{\psi}_j[\ ]\hat{\psi}_j^{\perp}$$

so that the polarization vector operators $\sigma_{\hat{x}}$ serve equally well to define the natural operator-valued charts over $\mathscr{P}_n\mathscr{C}$ as do the density operators $\hat{\psi}_{\hat{x}}$ themselves. One immediate advantage of using the $\sigma_{\hat{x}}$ representation is that the $\sigma_{\hat{x}}$ operators lie in $\mathscr{R}^i_{S0}$ along with the natural chart operators $X_j(\hat{x})$, and, as $\hat{x}$ ranges through the chart domain $\mathscr{P}^j_n\mathscr{C}$, $\sigma_{\hat{x}}$ ranges through an open set of a real $2n$-dimensional unit sphere in $\mathscr{R}^i_{S0}$. Taking $\sigma_j = \sigma_{\hat{x}_j}$ as a pole of this sphere, the corresponding operator P_j projects $\sigma_{\hat{x}}$ (and hence $\psi_{\hat{x}}$) onto the equatorial hyperplane of the sphere orthogonal to σ_j and allows us to use standard spherical polar angle coordinates to coordinatize the states $\hat{x} \in \mathscr{P}^j_n\mathscr{C}$. Such angle coordinates turn out to be the most directly measurable quantities characterizing the states in actual physical laboratories. We will see this in specific detail in the examples we need later.

3. QUANTUM THEORY IN PROJECTIVE SPACE

The standard assumptions of quantum theory, expressed in terms of complex projective space $\mathscr{P}\mathscr{C}$ [cf. Weyl (1931) and Hermann (1973), for example] make explicit use of a particular choice of metric with the corresponding definition of adjointness, so that the space $\mathscr{R}^i_S$ ($\mathscr{C}$-linear self-adjoint operators on the underlying real space) is well defined. Then we have the following:

(a) The pure states of a physical system that can be described by $\mathscr{P}\mathscr{C}$ (i.e., states with all "removable statistics" refined away) correspond to the elements $\hat{x} \in \mathscr{P}\mathscr{C}$ as realized by the projection operators $\hat{\psi}_{\hat{x}} \in \mathscr{R}^i_S$.

(b) Observables on the system correspond to arbitrary operators $A \in \mathscr{R}^i_S$, with the expectation value $\overline{A}_{\hat{x}}$ of the observable corresponding to A on the state corresponding to $\hat{x}$ given by

$$\overline{A}_{\hat{x}} = \mathrm{tr}_{\mathscr{C}}(A\hat{\psi}_{\hat{x}})$$

where $\mathrm{tr}_{\mathscr{C}} = 1/2$ trace is the trace function computed with respect to the complex field $\mathscr{C}$.

Expectation values on a statistical ensemble of pure states $\hat{x}_j$ with probability weights p_j—a mixed state W—are then computed in the standard way:

$$\bar{A}_W = \sum_j p_j \bar{A}_{\hat{x}_j} = \mathrm{tr}_{\mathscr{C}}\left(A \sum_j p_j \hat{\psi}_{\hat{x}_j} \right)$$

so that mixed states can be represented by density operators $W = \sum_j p_j \hat{\psi}_{\hat{x}_j} \in \mathscr{R}_S^i$, i.e., positive operators W such that $W^2 < W$ and $\mathrm{tr}_{\mathscr{C}}(W) = 1$. (The pure state case is then the specialization $W^2 = W = \hat{\psi}$.)

These standard assumptions result in a limitation on the possible symmetry transformations of a physical system to those transformations that correspond to operators U on the underlying real space such that $U^+ = U^{-1}$ and either $Ui = iU$ or $Ui = -iU$, i.e., unitary $\mathscr{C}$-linear or $\mathscr{C}$-antilinear operators, acting according to the prescriptions

$$A \to A' = UAU^{-1}, \qquad \hat{\psi}_{\hat{x}} \to \hat{\psi}'_{\hat{x}} = U\hat{\psi}_{\hat{x}}U^{-1}$$

since only these transformations preserve all expectation values *and* the $\mathscr{C}$-linear self-adjoint projective character of the operator realizations $\hat{\psi}_{\hat{x}}$ assumed for states $\hat{x} \in \mathscr{P}\mathscr{C}$. This limitation leads immediately to the standard supposition that only infinite-dimensional projective spaces are suitable for the representation of physical systems, since, for example, the restricted Lorentz group [i.e., $SL(2, \mathscr{C})$] is well established experimentally as a symmetry group of physical systems and, as is well known, there are no unitary or anti-unitary operator representations of this group except in infinite dimensions. A second argument adduced for the infinite-dimensionality of physically relevant $\mathscr{P}\mathscr{C}$ spaces is that the standard quantum definition of conjugate observables

$$QP - PQ \sim iI$$

cannot be realized except in an infinite-dimensional space, if i is understood as a scalar (i.e., commuting with all operators). This follows from the fact that in finite-dimensional spaces the trace function is defined for all operators, with $\mathrm{tr}(QP - PQ) = 0$ and $\mathrm{tr}(I) \neq 0$. Hence, only in infinite-dimensional spaces, where $\mathrm{tr}(I)$ is not defined, can we have $QP - PQ \sim I$. To demolish this second argument all we need to do is note that, once the nature of i as a complex structure operator on an underlying real space is understood, the relation $QP - PQ \sim iI = i$ is easily satisfied in any space of

complex dimension $n > 0$. For example, if $n = 1$, the real space underlying $\mathscr{P}_1\mathscr{C}$ is of real dimension $2(n+1) = 4$, and so i is an element of the real Dirac algebra of operators on R^4, with $QP - PQ \sim i$ satisfied in an infinite variety of ways (e.g., take $i = \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$, $Q = \gamma_0$, $P = \gamma_1\gamma_2\gamma_3$).

On the other hand, the valid group representation argument for infinite-dimensional projective space depends crucially on the assumption in standard formulations that only self-adjoint representations of states and observables be used. If we note that, for any $T \in GL(n+1, \mathscr{C}) \cup \overline{GL}(n+1, \mathscr{C}) \equiv GL'(n+1, \mathscr{C})$ (i.e., any invertible operator on the underlying real space that either commutes or anticommutes with the complex structure operator i), the transformations $A' = TAT^{-1}$, $\hat{\psi}'_{\hat{x}} = T\hat{\psi}_{\hat{x}}T^{-1}$ leave invariant both the expectation value formula and the $\mathscr{C}$-linear nature of the operators involved, as well as the projective character of $\hat{\psi}_{\hat{x}}$, we are led to suspect that all transformations defined by $\mathscr{C}$-linear or $\mathscr{C}$-antilinear operators in the group $GL(2(n+1), R)$ should be candidates for symmetries of a physical system describable by $\mathscr{P}_n\mathscr{C}$, and that something is not quite right in the dependence on metric implied by the self-adjoint restriction imposed by standard formulations of quantum theory.

If the point of view advocated above can be successfully adopted, then we note that we have an immediate termination of the extensive search presently going on for physical symmetry groups that contain both $SL(2, \mathscr{C})$ and the unitary groups as natural subgroups, since for $n > 0$ the group $GL(n+1, \mathscr{C}) = \mathscr{C}_0 \times SL(n+1, \mathscr{C})$ contains both $SL(2, \mathscr{C})$ and $U(n+1) = U(1) \times SU(n+1)$. For example, the appropriate group of symmetries for a $\mathscr{P}_1\mathscr{C}$ system would be exactly $GL(2, \mathscr{C}) = \mathscr{C}_0 \times SL(2, \mathscr{C})$, where $SL(2, \mathscr{C})$ decomposes into the 3-space rotations $SU(2, \mathscr{C})$ and the relativistic boosts $SL(2, \mathscr{C}) - SU(2, \mathscr{C})$. The only obstacle to the recognition of $GL'(n+1, \mathscr{C})$ as the required group of symmetries is the exclusive emphasis in standard formulations on unitary and antiunitary operators (metric-preserving operators) as the only suitable realizations of physical symmetries.

We note also that we have a solution to the problems associated with infinite-dimensional projective spaces in physical theory, since no such spaces would be needed as base spaces for physical systems; quantum field theory, of course would still be infinite dimensional, since it is the theory of cross sections of bundles over the base space $\mathscr{P}_n\mathscr{C}$ defined by a physical system, and such spaces of cross sections are naturally infinite dimensional, but this is a far more tractable problem than that of infinite-dimensionality for the base space itself. We can of course continue to embed everything automatically in infinite-dimensional space, as required by the standard formulation of quantum theory, and thus have unitary operator representations for SL$(2, \mathscr{C})$, but this procedure [known mathematically as "Koopmanism"—the translation of a finite-dimensional nonlinear problem

into a linear infinite-dimensional one; cf. Abraham and Marsden (1978), p. 140] seems more to obscure the fundamental nature of the problem than to resolve it.

A principle that generalizes standard quantum theory in such a way that it becomes free of the questionable dependence on adjoint and metric (and hence also free of the need for infinite-dimensional projective spaces) can now be stated as the *principle of quantum relativity* (QR):

> The laws of physics should be dependent only on the differentiable structures of the mathematical representation spaces used to express the laws, and should be invariant under the choice of equivalent metrics defining the differentiable structure.

We give this principle a relativistic designation simply because it seems to incorporate into quantum theory a lesson that relativity has been teaching for some time: the differentiable Euclidean space structures of Newtonian physics remain in relativistic physics, but equivalent observers can see very different length and angle measurements (metric properties).

As applied to our earlier formulation of quantum theory for a physical system with states describable by elements $\hat{x}$ in $\mathscr{P}_n\mathscr{C}$, QR requires no modification of the trace formula for expectation values, $\overline{A}_W = \mathrm{tr}_{\mathscr{C}}(AW)$, since the trace function is definable in purely linear algebraic differentiable terms with no reference to metric—the sum of the spectral values (points $\lambda \in \mathscr{C}$ such that $C_\lambda = AW - \lambda I$ is noninvertible). For later reference we note that the same property also holds of course for the determinant ($\equiv$ product of spectrum), with the obvious relation $e^{\mathrm{tr}(A)} = e^{\Sigma\lambda} = \prod e^{\lambda} = \det(e^A)$ for arbitrary operators A in finite dimensions, so that $\mathrm{tr}(A) = \ln\det(e^A)$.

However, the space $\mathscr{R}^i_S$ of $\mathscr{C}$-linear self-adjoint operators used in the standard formulations of quantum theory clearly does not have the invariance required by QR, and so it must be replaced. The necessary generalization is immediately at hand in the class $\mathscr{R}^i_{RD}$ of $\mathscr{C}$-linear *real diagonal* operators, defined as the class of operators having real spectrum and a complete set of eigenvectors, so that in the metric making the eigenvectors mutually orthogonal an operator in $\mathscr{R}^i_{RD}$ will be self-adjoint. Thus, an awkward but descriptive term for $\mathscr{R}^i_{RD}$ would be "self-adjointable" operators on $R^{2(n+1)}$, i.e., self-adjoint in some metric g differentiably equivalent to the standard Euclidean metric g_0 on $R^{2(n+1)}$, and so using the self-adjoint operators $\mathscr{R}^i_S$ relative to any fixed metric g we have

$$\mathscr{R}^i_{RD} = \left\{TAT^{-1}\colon\ A \in \mathscr{R}^i_S,\ T \in GL'(n+1, \mathscr{C})\right\}$$

since a transform defined by $T \in GL'(n+1, \mathscr{C})$ (similarity transform) leaves

invariant both spectrum and linear independence of eigenvectors. Now for an arbitrary $A' = TAT^{-1} \in \mathscr{R}^i_{RD}$ as above, with some initial metric g_0 defining $\mathscr{R}^i_S$, if v_j is any eigenvector of $A \in \mathscr{R}^i_S$, then $v'_j = Tv_j$ is obviously the corresponding eigenvector of A', and the metric g in which the basis $\{v'_j\}$ will be mutually orthogonal is straightforwardly computed as

$$g(x, y) = g_0(x, G_T y),\ x, y \in R^{2(n+1)}$$

where G_T is the positive invertible operator defined by $G_T = [TT^+]^{-1}$. Thus, if $\mathscr{H}^i_{g_0} = \{R^{2(n+1)}, h^i_0\}$ is the complex Hilbert space defined by the Hermitian metric $h^i_0(x, y) = g_0(x, y) + ig_0(ix, y)$, and $\mathscr{H}^i_g$ is the corresponding space for metric g, then T: $\mathscr{H}_{g_0} \to \mathscr{H}_g$ defines a *unitary* (or *antiunitary*) *transformation* of one Hilbert space onto another, both completely equivalent insofar as the differentiable structure of $\mathscr{P}_n\mathscr{C}$ is concerned. Since the transformations defined by $T \in GL'(n+1, \mathscr{C})$ all equally well preserve physical expectation values, they can (and will) be interpreted as symmetries of the physical system. The particular case $T \in U(n+1, \mathscr{C})$ just defines isometric transformations from $\mathscr{H}_{g_0}$ onto itself (unitary *operator* case) and should not be accorded the special status given to it by the usual formulation of quantum theory.

We emphasize, then, that to incorporate QR the only changes needed in the standard formulation of quantum theory as given earlier consist in everywhere replacing $\mathscr{R}^i_S$ with $\mathscr{R}^i_{RD}$, and using the correspondingly broadened understanding of physical symmetry transformation (generalized frame change). Thus, for example, $SL(2, \mathscr{C})$ appears as a natural unitary transformation symmetry group intrinsic to the differential geometry of the state space $\mathscr{P}_n\mathscr{C}$ postulated by quantum theory for any system with $n \geqslant 1$. It should also be clear now that we can take one further step if we allow the complex structure operator i itself to transform, $i \to i' = TiT^{-1}$. Then any $T \in GL[2(n+1), R]$ defines a unitary transformation T: $\mathscr{H}^i_{g_0} \to \mathscr{H}^{i'}_g$ leaving the underlying differential structure of $\mathscr{P}_n\mathscr{C}$ invariant, so that ultimately $GL(2(n+1), R)$ should be able to be seen as the full symmetry group for an n-dimensional complex projective system [this has been independently suggested in the case $n = 1$, in a purely gravitational context, by Kunstatter et al. (1983)]. We will investigate this in future work, but for now we need to establish answers to questions of interpretation by studying the more restricted $\mathscr{C}$-linear or antilinear ($i \to -i$) symmetries in their simplest nonstationary application, $\mathscr{P}_1\mathscr{C}$. Finally, it is interesting to note that the simplest space of all, $\mathscr{P}_0\mathscr{C}$ (single stationary state), has been proposed elsewhere in these same Conference Proceedings (Wootters, 1984) as the appropriate quantum model for the one known unique physical system—the universe as a whole. In light of the next section, we believe this proposal can be seen to fit well into the revision of quantum theory proposed here.

4. $P_1\mathscr{C}$ SYSTEMS

The quantum state space for the simplest possible physical systems exhibiting any change in expectation values (simplest nonstationary systems) is obviously $\mathscr{P}_1\mathscr{C}$, with underlying real space R^4 and operator space $R^4 \otimes R^4 = R^{16}$. Since this is simply the standard real Dirac algebra of operators, we will write $\mathscr{D}$ for R^{16} rather than $\mathscr{R}$. Choosing a complex structure operator $i \in \mathscr{D}$, is equivalent in standard physical notation to specifying $\gamma_5 \in \mathscr{D}$, and so we will identify $i = \gamma_5$, with the only defining properties $i^2 = \gamma_5^2 = -I \in \mathscr{D}$ and $i^+ = \gamma_5^+ = -i = -\gamma_5$, where the adjoint operation is defined by the usual Euclidean metric g_0 on R^4. Then, as in the general case given earlier, we construct $R_0^4 \equiv R^4 - \{0\}$, $\mathscr{C} \equiv \{aI + bi\} \subset \mathscr{D}$, $\mathscr{C}_0 = \mathscr{C} - \{0\} \subset \mathscr{D}$, and finally the space of orbits $\mathscr{P}_1\mathscr{C} = R_0^4/\mathscr{C}_0 = \{\hat{x} = \mathscr{C}_0 x\colon\ x \in R_0^4\}$ of the complex group $\mathscr{C}_0$ acting on R_0^4. Thus, since R^4 equipped with complex structure i has complex dimension 2 (i.e., $\{R^4, i\} \sim \mathscr{C}^2$), the manifold $\mathscr{P}_1\mathscr{C}$ has real dimension 2 and complex dimension 1.

An atlas of charts for the standard differential structure on $\mathscr{P}_1\mathscr{C}$ is then specified by choosing any two orbits $\hat{y}, \hat{y}^\perp \in \mathscr{P}_1\mathscr{C}$, orthogonal in the usual metric g_0 (equivalent to choosing orthogonal 2-planes in R_4), and using the two $\mathscr{C}$-linear g_0-self-adjoint projection operators $\hat{\psi}_{\hat{y}}$, $\hat{\psi}_{\hat{y}}^\perp = \hat{\psi}_{\hat{y}^\perp} = I - \hat{\psi}_{\hat{y}} \in \mathscr{D}^i$ onto these 2-planes to construct two chart domains:

$$\mathscr{P}_1^{\hat{y}}\mathscr{C} = \left\{\hat{x} \in \mathscr{P}_1\mathscr{C}\colon\ \hat{\psi}_{\hat{y}}\hat{\psi}_{\hat{x}} \neq 0\right\}$$

$$\mathscr{P}_1^{\hat{y}^\perp}\mathscr{C} = \left\{\hat{x} \in \mathscr{P}_1\mathscr{C}\colon\ \hat{\psi}_{\hat{y}}^\perp\hat{\psi}_{\hat{x}} \neq 0\right\}$$

(In the special case of $\mathscr{P}_1\mathscr{C}$, of course, both $\hat{\psi}$ and $\hat{\psi}^\perp = I - \hat{\psi}$ represent elements of $\mathscr{P}_1\mathscr{C}$.) The required atlas of charts consists, then, of the two operator-valued mappings onto open sets in the real two-dimensional subspaces $\mathscr{D}_{\hat{y}}^i, \mathscr{D}_{\hat{y}^\perp}^i \subset \mathscr{D}_{S0}^i$ defined by

$$X_{\hat{y}}\colon\ \mathscr{P}_1^{\hat{y}}\mathscr{C} \to \mathscr{D}_{\hat{y}}^i$$

$$\hat{x} \to X_{\hat{y}}(\hat{x}) = P_{\hat{y}} \cdot \hat{\psi}_{\hat{x}}$$

and

$$X_{\hat{y}^\perp}\colon\ \mathscr{P}_1^{\hat{y}^\perp}\mathscr{C} \to \mathscr{D}_{\hat{y}^\perp}^i$$

$$\hat{x} \to X_{\hat{y}^\perp}(\hat{x}) = P_{\hat{y}^\perp} \cdot \hat{\psi}_{\hat{x}}$$

where, in this special case of $\mathscr{P}_1\mathscr{C}$,

$$P_{\hat{y}} = P_{\hat{y}^\perp} = \hat{\psi}_{\hat{y}}^\perp[\ ]\hat{\psi}_{\hat{y}} + \hat{\psi}_{\hat{y}}[\ ]\hat{\psi}_{\hat{y}}^\perp$$

is the projection operator on the space $\mathscr{D}^i$ (projecting into the subspace $\mathscr{D}^i_{S0}$ of $\mathscr{C}$-linear trace-zero self-adjoint operators in the Dirac algebra $\mathscr{D}$). Thus the only difference between the two charts $X_{\hat{y}}$ and $X_{\hat{y}^\perp}$ lies in the different domains required to guarantee that the mappings are one-to-one:

$$\mathscr{P}_1^{\hat{y}}\mathscr{C} = \mathscr{P}_1\mathscr{C} - \{\hat{y}^\perp\}$$

$$\mathscr{P}_1^{\hat{y}^\perp}\mathscr{C} = \mathscr{P}_1\mathscr{C} - \{\hat{y}\}$$

and the two charts agree on the intersection $\mathscr{P}_1\mathscr{C} - \{\hat{y}, \hat{y}^\perp\}$. Choosing, say, Euclidean coordinate axes in the real 2-space $\mathscr{D}^i_{\hat{y}} = \mathscr{D}^i_{\hat{y}^\perp}$ gives one standard type of coordinatization for $\mathscr{P}_1\mathscr{C}$.

The spin vectors or polarization vectors $\sigma_{\hat{x}} \in \mathscr{D}^i_{S0}$ corresponding to the projections $\hat{\psi}_{\hat{x}}$ representing states of a $\mathscr{P}_1\mathscr{C}$ system are particularly simple; from our earlier definition we have

$$\sigma_{\hat{x}} = \tfrac{1}{2}\left(\hat{\psi}_{\hat{x}} - \hat{\psi}^\perp_{\hat{x}}\right) = \hat{\psi}_{\hat{x}} - \tfrac{1}{2}I$$

with $\sigma_{\hat{x}}^2 = (1/4)I \in \mathscr{D}$, so that $\mathrm{tr}(\sigma_{\hat{x}}^2) = 1$, while $\mathrm{tr}_{\mathscr{C}}(\sigma_{\hat{x}}^2) = 1/2$. Thus, in terms of the real Dirac algebra structure defining the differential geometry of $\mathscr{P}_1\mathscr{C}$, $\sigma_{\hat{x}}$ is a family of unit vectors sweeping out a unit 2-sphere in the real 3-space $\mathscr{D}^i_{S0}$ of $\mathscr{C}$-linear self-adjoint trace-0 operators in $\mathscr{D}$, with self-adjointness defined in terms of the original Euclidean metric chosen on R^4. Choosing some $\sigma_{\hat{y}}$ to define a pole of the sphere (z axis) and using $P_{\hat{y}}$ to project $\sigma_{\hat{x}}$ (and hence $\hat{\psi}_{\hat{x}}$, since $P_{\hat{y}}\sigma_{\hat{x}} = P_{\hat{y}}\hat{\psi}_{\hat{x}}$) onto the equatorial plane orthogonal to $\sigma_{\hat{y}}$ in $\mathscr{D}^i_{S0}$, we can use standard polar angle coordinates θ, φ on the sphere, relative to $\sigma_{\hat{y}}$ and some $\sigma_{\hat{y}'}$ in the plane, to coordinatize $\mathscr{P}_1\mathscr{C}$ systems. We note, however, that general observables A on a $\mathscr{P}_1\mathscr{C}$ system, including the density operators $\hat{\psi}_{\hat{x}} = \sigma_{\hat{x}} + (1/2)I$, live in the full real 4-space $\mathscr{D}^i_S$ of $\mathscr{C}$-linear self-adjoint operators in the Dirac algebra, spanned by $I \in \mathscr{D}$ and the trace-zero operators $\sigma_{\hat{x}} \in \mathscr{D}^i_{S0}$.

Now we come to an important point for interpretational purposes: the observables $\sigma_{\hat{x}} = (1/2)(\hat{\psi}_{\hat{x}} - \hat{\psi}^\perp_{\hat{x}})$ on a $\mathscr{P}_1\mathscr{C}$ system have an experimentally well-established meaning as the intrinsic angular momentum vectors corresponding to the states of a spin-1/2 system, and an essential part of this interpretation lies in the identification of expectation values such as $\mathrm{tr}_{\mathscr{C}}(\sigma_{\hat{y}}\sigma_{\hat{x}}) = \mathrm{tr}_{\mathscr{C}}(\sigma_{\hat{y}}\psi_{\hat{x}})$ as the components of the spin vector along the various possible directions of the observable 3-space of ordinary physical experience (Schiff, 1968, p. 382). Hence, if we wish to preserve this standard interpretation, we must identify the Dirac algebra 3-space $\mathscr{D}^i_{S0}$ with the 3-space of ordinary physical experience, and, more generally, the Dirac algebra 4-space of observables $\mathscr{D}^i_S$ with the observables defining relativistic

physics. We will see shortly that this results in the standard interpretation of elements of the Dirac algebra as used, say, in the Dirac equation. We can reverse this line of argument, also, and see that, to the extent that an observer interacts directly only with $\mathscr{P}_1\mathscr{C}$ systems, the observer would find himself living in a world representable by elements of the Dirac algebra precisely because the observed systems live in the mathematical space $\mathscr{P}_1\mathscr{C}$.

How broadly, however, can we realistically expect such a mathematically simple model as $\mathscr{P}_1\mathscr{C}$ to apply? With some understanding of the richness of the mathematical structures involved—underlying real 4-space with 16-dimensional operator space needed to define complex structure and differential geometry—we can well begin to suspect much wider application than a naive first reaction would suggest, but certainly (by definition) the model applies to a single spin-1/2 particle as long as its interactions with the rest of the universe can be described by potentials and fields completely definable within the mathematical structure available over $\mathscr{P}_1\mathscr{C}$. Then, by the standard conventions and methods for dealing with N particle systems in terms of product spaces, we can extend the coverage of $\mathscr{P}_1\mathscr{C}$ to such systems, again under the proviso that the effects on any individual particle can be modeled within the differentiable structure of a single $\mathscr{P}_1\mathscr{C}$ space (just as, for example, a classical system of N particles with $3N$-dimensional configuration space is modeled "in" a single 3-space under the analog of the above proviso). Is there an even broader applicability of $\mathscr{P}_1\mathscr{C}$, though, say to systems that might intrinsically require a very high-dimensional projective space for their complete description? There is, if we can introduce into quantum theory the same sort of scale invariance assumed for classical theory. Here, "scale invariance" is being used to mean the property that allows us to treat macroscopic systems (even cosmological structures such as clusters of galaxies) as simple mass points moving in effective gravitational and electromagnetic fields if we can take a sufficiently far removed viewpoint so that the complicated internal kinematics and dynamics of the system become unobservable and ignorable. Then the appropriate measurable quantities are "thermodynamic" observables such as total mass, total luminosity, etc.

We can introduce this type of scale invariance into quantum theory by noting that an arbitrary density operator $W=\sum_j \lambda_j \hat{\psi}_j$ (representing a possibly highly complicated mixture of pure states from an arbitrary-dimensional projective space), as a positive operator on a Hilbert space $\mathscr{H}$, defines a positive square root operator $W^{1/2}$ that is a unit length vector in $\mathscr{H}\mathscr{S}$ (the space of trace square-normable operators on $\mathscr{H}$), i.e., $\mathrm{tr}(W)=1 \Rightarrow \mathrm{tr}(W^{1/2}W^{1/2})=1 \Rightarrow W^{1/2}\in\mathscr{H}\mathscr{S}$. Then $\hat{\psi}_W \equiv W^{1/2}\otimes W^{1/2}$ defines a one-dimensional projection operator on the complex Hilbert space $\mathscr{H}\mathscr{S}$, and hence an element $\hat{W}\in\mathscr{P}_{\mathscr{H}\mathscr{S}}\mathscr{C}$, i.e., an element of the complex projective

space of $\mathscr{H}\mathscr{S}$. Now suppose that, as W evolves, $\hat{W}$ stays in a one-dimensional complex projective submanifold $\mathscr{P}_1\mathscr{C} \subset \mathscr{P}_{\mathscr{H}\mathscr{S}}\mathscr{C}$; then the $\mathscr{P}_1\mathscr{C}$ model applies as before, and we can define observables on W by elements in the Dirac algebra corresponding to $\mathscr{P}_1\mathscr{C}$. For example, if T is the operator, on the original Hilbert space $\mathscr{H}$, that represents kinetic energy, so that $\mathrm{tr}(TW) = \sum_j \lambda_j \mathrm{tr}(T\hat{\psi}_j) = \overline{T}_W$ gives the average or expected value of kinetic energy for the system represented by W, then the "superoperator" $\mathbf{T}$ on $\mathscr{H}\mathscr{S}$, defined by left-multiplication, i.e., $\mathbf{T}W^{1/2} \equiv TW^{1/2} \in \mathscr{H}\mathscr{S}$, gives the operator on $\mathscr{H}\mathscr{S}$ defining average internal kinetic energy (*temperature*) for $\hat{W}$, with

$$\mathrm{tr}_{\mathscr{H}\mathscr{S}}(\mathbf{T}\hat{\psi}_W) = \mathrm{tr}_{\mathscr{H}\mathscr{S}}(\mathbf{T}[W^{1/2} \otimes W^{1/2}]) = \mathrm{tr}_{\mathscr{H}}(W^{1/2}[\mathbf{T}W^{1/2}])$$

$$= \mathrm{tr}_{\mathscr{H}}(W^{1/2}TW^{1/2}) = \mathrm{tr}_{\mathscr{H}}(TW) = \overline{T}_W$$

When the evolution of $\hat{\psi}_W$ is restricted as described above to a $\mathscr{P}_1\mathscr{C}$ submanifold, letting P_W mean the projection onto the resulting complex 2-space in $\mathscr{H}\mathscr{S}$, we can obviously replace $\mathbf{T}$ by the Dirac algebra operator $P_W\mathbf{T}P_W$ over $\mathscr{P}_1\mathscr{C}$ without loss of information (the compression of $\mathbf{T}$ to the 2-space). Since arbitrary observables A on the original Hilbert space $\mathscr{H}$ define such left-multiplication operators $\mathbf{A}$ and the corresponding compression $P\mathbf{A}P$, we see that all observable information on $\hat{W} \in \mathscr{P}_1\mathscr{C}$ is contained in the corresponding Dirac algebra of operators, and we might well begin to suspect that the origin of relativistic physics lies in the fact that our ordinary life experience is built up from just such statistical observations of $\mathscr{P}_1\mathscr{C}$ systems $\hat{W}$. In any case, our working hypothesis will be that the mathematics of $\mathscr{P}_2\mathscr{C}$, with corresponding symmetry groups $SU(3) \subset SL(3, \mathscr{C}) \subset GL(6, R)$, and all higher-dimensional geometries, will not be needed in quantum physics (which we of course understand as encompassing all of physics) until we encounter systems exhibiting extreme properties relative to the standards of ordinary macroscopic experience, such as extremely high temperatures, energy densities, etc. Such systems would of course show up as singularities of any mathematical model that tries to fit them into the mold appropriate only for the lower-dimensional 4-space mathematics over $\mathscr{P}_1\mathscr{C}$.

The "superoperator" formulation given above is, of course, not new and has been tacitly understood in standard applications since the beginning of quantum theory, e.g., in spectroscopy, where today's pure states, as determined by available line spectra, become tomorrow's statistical mixtures as more refined instrumentation reveals line splitting. The introduction of the scale invariant $\hat{\psi}_W$ model simply incorporates into quantum theory the flexibility necessary to handle our inability to define operationally "ulti-

mate pure states," and allows us to treat systems of arbitrary scale or complexity in a unified formalism. Thus we can always assume that quantum theory is to be formulated in the projective space (ray space) of a complex Hilbert space of *operators* of indeterminate but probably large dimension and our working hypothesis comes down to the assumption that "ordinary experience" is defined by systems $\hat{W}$ confined to the simplest projective submanifolds of the overall ray space. With this understanding, we will continue to write $\hat{x} \in \mathscr{P}_1\mathscr{C}$, $\hat{\psi}_{\hat{x}}$, etc., allowing for the fact that $\hat{x}$ may well be a projective element $\hat{W}$ corresponding to a density operator W of large dimensionality (i.e., dimension of range space = rank).

To the extent that the physical measuring apparatus available was limited to the type capable of registering only quantum expectation values (examples: hand touching warm object or thermometer giving a more refined measure of temperature, but still only registering an average over many transferals of kinetic energy; eye or photometer registering average intensity; etc.), one might well expect that a careful study of the evidence would result in a theory of observables as real functions $\overline{A}$ on a system space of elements $\hat{x}$, where the function values $\overline{A}_{\hat{x}}$ could be computed from a quadratic form $\overline{A}_{\hat{x}} = \mathrm{tr}_{\mathscr{C}}(A\hat{\psi}_{\hat{x}})$ over a real four-space (our $\mathscr{D}_S^i$) if correct vector representatives

$$A = A^I I + A^j\sigma_j = A^I I + A^j\gamma_j\gamma_0 \in \mathscr{D}_S^i$$

$$\hat{\psi} = \tfrac{1}{2}I + \psi^j\sigma_j = \tfrac{1}{2}I + \psi^j\gamma_j\gamma_0 = \tfrac{1}{2}I + \sigma_{\hat{x}} \in \mathscr{D}_S^i$$

were assigned to observables and systems in the four-space. (We will start using freely both the standard summation convention and the usual notation for orthonormal bases $\sigma_j = \gamma_j\gamma_0 \in \mathscr{D}_{S0}^i$ in the 3-space of trace-0 self-adjoint Dirac operators.) A fundamental distinction between vectors in the 3-space $\mathscr{D}_{S0}^i$ and the I component would become evident as soon as the natural symmetry group for $\mathscr{P}_1\mathscr{C}$ showed up sufficiently clearly in the experimental evidence and an attempt was made to treat the symmetries as frame transformations of the 4-space, since for arbitrary $A \in \mathscr{D}_S^i$, $T \in GL'(2, \mathscr{C})$:

$$A' = TAT^{-1} = A^I I + A^j T\sigma_j T^{-1}$$

that is, the I component behaves as a scalar invariant under all such frame changes, with

$$\mathrm{tr}_{\mathscr{C}}\left(A'\hat{\psi}'_{\hat{x}}\right) = A^I + \mathrm{tr}_{\mathscr{C}}\left(\mathbf{A}'\boldsymbol{\sigma}'_{\hat{x}}\right)$$

where we adopt the standard vector designation **A** for the 3-space elements in $\mathscr{D}^{i}_{S0}$. Moreover, given the standard interpretation of frame changes in physics as specifying possible kinematics when applied separately to one or the other vector in the above formula—$\mathrm{tr}_{\mathscr{C}}(A'\hat{\psi}_{\hat{x}}) = \mathrm{tr}_{\mathscr{C}}(TAT^{-1}\hat{\psi}_{\hat{x}})$ would represent observations made from a transformed observer frame or lab relative to a fixed system ("passive" view of transformations or quantum theoretical Heisenberg picture), while $\mathrm{tr}_{\mathscr{C}}(A\hat{\psi}''_{\hat{x}}) = \mathrm{tr}_{\mathscr{C}}(AT^{-1}\hat{\psi}_{\hat{x}}T)$ would represent the same situation interpreted as observations made from a fixed lab frame on an inversely transformed, i.e., boosted and/or rotated, system ("active" view or quantum theoretical Schroedinger picture)—one would find that I components give kinematical constants of the motion as well, since

$$\mathrm{tr}_{\mathscr{C}}(A'\hat{\psi}_{\hat{x}}) = A^{I} + \mathrm{tr}_{\mathscr{C}}(\mathbf{A}'\sigma_{\hat{\psi}_{\hat{x}}})$$

$$= \mathrm{tr}_{\mathscr{C}}(A\hat{\psi}''_{\hat{x}}) = A^{I} + \mathrm{tr}_{\mathscr{C}}(\mathbf{A}\sigma''_{\hat{\psi}_{\hat{x}}})$$

Thus, given the mathematical prominence of quadratic invariance theory at the beginning of this century when the earliest such invariant, c, was recognized and incorporated into physical theory along with invariance under $SL(2,\mathscr{C})$ frame changes in the form of Lorentz transformations, we can see that it would not be unnatural to look for a new quadratic form on the underlying 4-space representing observables and systems that would intrinsically express the invariant character of the I components and allow computation of these invariants by the simple rearrangement

$$A^{I} = \mathrm{tr}_{\mathscr{C}}(A\hat{\psi}_{\hat{x}}) - \mathrm{tr}_{\mathscr{C}}(\mathbf{A}\sigma_{\hat{\psi}_{\hat{x}}})$$

Since the fundamental work of W.K. Clifford in the last century, it has been recognized that the natural means of expressing and studying such scalar invariance lies in the construction of an algebra intrinsically associated to the desired quadratic form, and in this way we can see that the full 16-dimensional Dirac algebra, as the Clifford algebra associated with the required pseudometric, would be discovered. For a physical treatment of Clifford algebra and particularly the relativistic Dirac algebra, see Hestenes (1966) or Casanova (1976).

In terms of pure $\mathscr{P}_1\mathscr{C}$ quantum theory, of course, the Dirac Clifford algebra is simply the easiest way to incorporate the invariance of differentiable structure under transformations to equivalent metrics required by the QR principle, and first shows up as an essential component of the differentiable geometry of $\mathscr{P}_1\mathscr{C}$ as the algebra from which the complex structure

must be selected. The above construction of idealized history (based on the question "What if the quadratic nature of the quantum expectation value formula $\mathrm{tr}_{\mathscr{C}}(A\hat{\psi}_{\hat{x}})$ became evident in physical data before the full structure of quantum theory was understood?") might be taken simply as a parable illustrating the fact that an interpretation of classical special relativistic physics as a theory of "thermodynamic," i.e., statistical, observables on $\mathscr{P}_1\mathscr{C}$ systems is viable and widely applicable, in terms of the scale-invariant quantum theory presented earlier. However, we prefer to try a more literal interpretation of the idealized history, since, granted the basic assumption of this paper that quantum theory is the most fundamental theory of physical reality available to date, something like the above reconstruction must have occurred, and relativistic physics (as all physics) should have a natural origin from quantum physics. We incorporate all this by adopting a guiding principle that we will refer to as the *QR principle of economy*:

> Make only the minimal changes in standard mathematical formalism and interpretation of both relativity and quantum theory consistent with the QR principle and the resulting finite-dimensional projective space structures.

Since all elements of the Dirac algebra have standardly accepted interpretations as real laboratory-definable quantities shared in common by both quantum physics and relativistic physics, we thus keep this shared interpretation by assuming that macroscopic observers (i.e., human-scale observers) order their experience of physical systems in terms of some macroscopically observable parameter t, which allows the assignment of a Dirac algebra vector trajectory $\gamma_t \in \mathscr{D}$, with $\gamma_t^2 = I$, to the observer's lab frame in such a way that γ_t serves to keep track of the proper definition of adjointness (and hence proper 3-space metric) at each observer instant as arbitrary families $T_t \in SL(2,\mathscr{C})$ are used to generate the observed evolution of systems and/or observables. Specifically, starting with an initial γ_0 and the metric g_0 on the underlying R^4 that makes γ_0 self-adjoint, we note that arbitrary families $T_t \in SL(2,\mathscr{C})$ can be written in standard (possibly t-dependent) Hamiltonian form as $T_t = e^{\iota H_t}$, where the bivector $H_t = z_t^j\gamma_j\gamma_0 = (x_t^j + iy_t^j)\gamma_j\gamma_0$ splits into a self-adjoint part $x^j\gamma_j\gamma_0$ (generator of relativistic boost) and an anti-self-adjoint part (generator of 3-space rotation). Defining $\gamma_t = e^{\iota H_t}\gamma_0 e^{-\iota H_t}$, it is straightforward to check that, if $\bar{A}$ represents the adjoint of A in the g_0 metric, then $A^+ = \gamma_t\bar{A}\gamma_t$ gives the proper definition of adjointness in the new metric g_t in which γ_t is self-adjoint. Thus the standard relativistic assignment of unit timelike Dirac 4-vectors to define instantaneous direction of possible observer lab frame trajectories (world lines) turns out, in terms of pure $\mathscr{P}_1\mathscr{C}$ quantum theory, to be a way of keeping track of the metric changes allowed by the QR principle, as systems and/or observables vary arbitrarily.

Thus, to each such observer frame we can assign "proper" observable and state representations (self-adjoint in the metric making γ_t self-adjoint) and "improper" non-self-adjoint representations which must be used to compute expectation values when the observed system $\hat{x}$ and observer frame are boosted relative to one another and it becomes evident that any formulation of quantum theory that allows only a fixed metric, with corresponding fixed meaning for self-adjointness, and only unitary operators as frame symmetries and generators of the motion, is an intrinsically nonrelativistic formulation in the sense that, to introduce relativistic physics, we must embed everything in infinite-dimensional space and introduce the Dirac algebra of observables in a purely *ad hoc* way.

An additional advantage of the finite-dimensional $\mathscr{P}_1\mathscr{C}$ formulation is that a natural distinction between QR kinematics and QR dynamics appears. Assuming for the moment the Schroedinger picture, with the observer considering his observing lab and timelike frame vector γ_0 fixed, and the Dirac spinor $\hat{\psi}_{\hat{x}}(t) = (1/2)I + \sigma_{\hat{x}}(t)$ representing system states $\hat{x}$ evolving according to $\hat{\psi}_{\hat{x}}(t) = e^{tH}\hat{\psi}_{\hat{x}}e^{-tH}$, where $\hat{\psi}_{\hat{x}} = (1/2)I + \sigma_{\hat{x}}$ is some fixed Dirac spinor with $\sigma_{\hat{x}} = \psi^j(\hat{x})\gamma_j\gamma_0$, we see that there is a natural splitting of Dirac algebra Hamiltonians H into two parts H_0 (changing $\hat{\psi}$ but not $\hat{x}$) and $H_1 = H - H_0$ (changing $\hat{x}$). The requirement on H_0 is that it have $\hat{x} \subset R^4$ as an eigenspace, but it need not have $\hat{x}^\perp$ (as determined in the metric g_0 corresponding to γ_0) as an eigenspace. Thus H_0 would cause $\hat{\psi}_{\hat{x}}(t)$ to vary through non-self-adjoint projective representations of the same fixed state $\hat{x} \in \mathscr{P}_1\mathscr{C}$ relative to γ_0, since $\hat{\psi}_{\hat{x}}(t)$ would have fixed range $\hat{x}$ but varying kernel space $\hat{x}^\perp$ in R^4. On the other hand, H_1 would be the part of the Dirac Hamiltonian not having $\hat{x}$ as an eigenspace, and hence causing actual evolution of the state $\hat{x}_t \in \mathscr{P}_1\mathscr{C}$. The splitting of course would not be unique, but only up to a gauge operator H having the given initial $\hat{x} \in \mathscr{P}_1\mathscr{C}$ as an eigenspace. Thus H_0 corresponds to the free particle Dirac Hamiltonian and its four real eigenspaces $\hat{x}, i\hat{x}, \hat{x}', i\hat{x}' \subset R^4$ correspond to stationary states, each still capable of being observed under arbitrary kinematical frame changes defined by all non-self-adjoint projective representations $\hat{\psi}_{\hat{x}}$, and H_1 gives the part of the Dirac Hamiltonian representing the electromagnetic vector potential (with real and imaginary parts given their standard interpretation), conceived of as corresponding to interactions of the system with the rest of the universe causing dynamical evolution of the system $\hat{x}_t \in \mathscr{P}_1\mathscr{C}$. Fourier analysis of the free particle eigenoperator case $\exp(tH_0)\psi_{\hat{x}}\exp(-tH_0) = \exp(t[\vec{H}_0 - \bar{H}_0])\hat{\psi}_{\hat{x}}$, where $\vec{H}_0$ and $\bar{H}_0$ are left-and-right-multiplication superoperators defined by H_0 and the various component operators $\overrightarrow{\gamma_j\gamma_0} - \overleftarrow{\gamma_j\gamma_0}$, $i[\overrightarrow{\gamma_j\gamma_0} - \overleftarrow{\gamma_j\gamma_0}] = [\overrightarrow{\gamma_k\gamma_l} - \overleftarrow{\gamma_k\gamma_l}]$ are regarded as defining linear momentum and angular momentum lab axes, respectively, results in the usual quantum relativistic conjugacy definitions if

we wish to work in relativistic event space instead of energy-momentum space, with $t = x_0$ transforming to a scalar multiple of differentiation along the energy axis $\vec{H} - \bar{H}$, x^j transforming to a scalar multiple of differentiation along the jth momentum axis $\gamma_j\gamma_0 - \overleftarrow{\gamma_j\gamma_0}$, etc. Finally, we note that in the Dirac superoperator formalism the spinor $\hat{\psi}_{\hat{x}} = (1/2)I + \sigma_{\hat{x}}$ takes its standard 4-component "column" form as a complex vector on which $\exp(t[\vec{H}_0 - \bar{H}_0])$ acts (as an element of $\mathscr{D} = R^{16}$ commuting with i, $\hat{\psi}_{\hat{x}}$ is a 4 complex component matrix, which gets arrayed as a column in the superoperator formalism).

All this development fits rather naturally into the field theory of the next section, but we can see that, with the introduction of an interaction term H_1 into the Dirac Hamiltonian, field theory becomes particularly important since then we are essentially discussing possible assignments of Dirac-algebra-valued cross sections $\hat{\psi}_{\hat{x}}(t) = \hat{\psi}_{\hat{x}_t}$ over elements $\hat{x}_t$ varying through the manifold $\mathscr{P}_1\mathscr{C}$, as opposed to the fixed state considerations of the free particle case.

5. FIELD THEORY OVER $\mathscr{P}_1\mathscr{C}$

The main advantage of finite-dimensional projective space quantum theory is precisely that we can use completely finite-dimensional methods over a compact base manifold $\mathscr{P}_n\mathscr{C}$; in the case of $\mathscr{P}_1\mathscr{C}$ we have the additional feature that the standard relativistic mathematics of Lorentz transformations and the 16-dimensional Dirac algebra of associated geometric quantities appear uniquely and naturally as the means for defining and studying the real and complex differentiable structure over the manifold. On the basis of the principle of economy, we then associate the relativistic Dirac algebra over $\mathscr{P}_1\mathscr{C}$ with the physically known occurrences of such structure (otherwise, on our assumption that quantum theory is the most fundamental available theory, we would be compelled to look for a separate *nonrelativistic* appearance of Lorentz invariance and the associated mathematical structure in the physical data relating to spin-1/2, i.e., $\mathscr{P}_1\mathscr{C}$, systems), and so we are led to try the hypothesis that our world of physical experience exhibits its relativistic character at a wide variety of scales, down to the nucleus and up to neutron stars, precisely because our experience at these different scales consists primarily of statistical observations on $\mathscr{P}_1\mathscr{C}$ quantum systems.

As currently understood in modern physical theory, the most general structure available for handling physically relevant differential geometry and symmetries is the mathematical language of vector bundles associated to a principal bundle over a manifold—the theory of gauge fields in physics.

When we look for such structures over complex projective space, we find that the definition of $\mathscr{P}_n\mathscr{C}$ involves a natural or *canonical principal bundle* $\hat{\pi}$ with $R_0^{2(n+1)}$ as bundle space and $\mathscr{C}_0$ as gauge group. Specifically, in the case of $\mathscr{P}_1\mathscr{C}$ the principal bundle structure is displayed by the mapping

$$\hat{\pi}\colon R_0^4 \overset{\mathscr{C}_0}{\to} \mathscr{P}_1\mathscr{C} \equiv R_0^4/\mathscr{C}_0$$

$$x \to \hat{\pi}(x) = \hat{x} = \mathscr{C}_0 x$$

where $\hat{\pi}$ is the *bundle projection* from the *bundle space* R_0^4 onto the *base space* $\mathscr{P}_1\mathscr{C}$, and the complex group $\mathscr{C}_0$ is the *gauge group* (invariance group) of the bundle, with elements x in a given orbit $\hat{x}$ of the group making up the fiber in R_0^4 over $\hat{x}$ as a base point of the bundle [see, for example, Bleecker (1981) for general mathematical treatment with physical applications].

Typical atlases of charts, defining the differentiable structure of R_0^4 as bundle space of $\hat{\pi}$, are constructed as follows: if $X_1 = X_{\hat{y}}$, $X_2 = X_{\hat{y}^\perp}$, are standard charts for $\mathscr{P}_1\mathscr{C}$ as specified earlier, with respective domains $\mathscr{U}_1 = \mathscr{P}_1\mathscr{C} - \{\hat{y}^\perp\}$, $\mathscr{U}_2 = \mathscr{P}_1\mathscr{C} - \{\hat{y}\}$, let x_1 and x_2 be *local sections* on these domains, i.e., for $n = 1, 2$,

$$x_n\colon \mathscr{U}_n \to R_n^4 = \hat{\pi}^{-1}(\mathscr{U}_n) \subset R_0^4$$

$$\hat{x} \to x_n(\hat{x}) \in \hat{x}$$

are suitably differentiable mappings into the usual differentiable structure on R^4 so that the real-valued mappings ρ_n and α_n on $\hat{\pi}^{-1}(\mathscr{U}_n)$, given by

$$x = \exp[\rho_n(x) + i\alpha_n(x)]\, x_n \circ \hat{\pi}(x)$$

are sufficiently differentiable. (We purposely leave the degree of differentiability required for physical theory unspecified, but of course, for concreteness, we could require C^∞ if desired; to avoid technical complexities we prefer to follow the standard physical practice, and in what follows we simply assume that all structures have sufficient differentiability for the purposes at hand.) Then, on each of the two chart neighborhoods $R_n^4 = \hat{\pi}^{-1}(\mathscr{U}_n)$, $n = 1, 2$, covering the bundle space R_0^4, a choice of chart X_n for the neighborhood $\mathscr{U}_n$ of the base space $\mathscr{P}_1\mathscr{C}$ and a choice of local section x_n defining standard logarithmic coordinates ρ_n, α_n, for the fiber spaces ($\sim \mathscr{C}_0$) over $\mathscr{P}_1\mathscr{C}$ specify completely a chart for the four-dimensional bundle space R_0^4. In terms of such a pair $(X_n \circ \hat{\pi},\ x_n \circ \hat{\pi})$ over $R_n^4 \subset R_0^4$, natural coordinatizations for the four-dimensional manifold R_0^4, convenient

for different purposes, are given by

$$x \to (\theta_n(x), \varphi_n(x), \rho_n(x), \alpha_n(x))$$

$$x \to (\theta_n(x), \varphi_n(x), \rho_n(x), i\alpha_n(x))$$

$$x \to \left(\theta_n(x), \varphi_n(x), e^{\rho_n(x)}, e^{i\alpha_n(x)}\right)$$

where, in all cases we have abbreviated $\theta_n(x) = \theta_n(\hat{\pi}(x))$, $\varphi_n(x) = \varphi_n(\hat{\pi}(x))$, for the standard lab polar angle coordinates of the spin vector $\sigma_{\hat{\pi}(x)}$ characterizing the $\mathscr{P}_1\mathscr{C}$ state $\hat{x} = \hat{\pi}(x)$, as defined earlier.

Because of the Abelian nature of the gauge group $\mathscr{C}_0$ involved in the canonical principal bundle construction of $\mathscr{P}_1\mathscr{C}$, the distinction between the principal bundle and its simplest associated vector bundles is much less pronounced than would be the case for a non-Abelian gauge group, since there is no difference between right and left actions in the Abelian case. Thus, the simplest type of vector bundle naturally associated with $\mathscr{P}_1\mathscr{C}$ (*canonical line bundle*) is only trivially different from the principal bundle itself, and is constructed by adding the zero point to each fiber $\hat{x}$, so that the fibers of the associated bundle $\hat{x} \cup \{0\} = \{\mathscr{C}_{\hat{x}}\}$ are full real 2-planes in R^4 or complex 1-spaces in $\mathscr{C}^2$, with 0 as a distinguished invariant point under the action of $\mathscr{C}_0$. The construction of more general vector bundles associated to $\mathscr{P}_1\mathscr{C}$ proceeds analogously, with complex Hilbert spaces $\mathscr{H}$ of various dimensions chosen as fiber, and, of course, with the action of $\mathscr{C}_0$ on $\mathscr{H}$ defined by the complex structure of $\mathscr{H}$.

Using the standard physical interpretation of cross sections of such associated vector bundles over $\mathscr{P}_1\mathscr{C}$ as "particle fields" describing particle-like quantum attributes associated with possible internal degrees of freedom of the basic physical states being modeled, we see that everything we have done earlier—including the scale-invariant $W^{1/2}$ construction of vector representatives for $\mathscr{P}_1\mathscr{C}$ systems with many dimensions of internal complexity—can be expressed in the language of associated bundles over $\mathscr{P}_1\mathscr{C}$ as an "external coordinate space," i.e., base space, provided only that the fundamental system states $\hat{x}$ can be regarded as confined to such a simple projective space. It should also be evident, once we note the fact that various representations of the gauge group $\mathscr{C}_0$ (including the adjoint representation, in this case the trivial representation $C \to I$) can be used to define the action on the associated fibers, that even our basic construction of Dirac algebra representations of states $\hat{\psi}_{\hat{x}}, \sigma_{\hat{x}}, X_n(\hat{x})$, etc., can be fitted into the pattern of the associated bundle construction.

In terms of the principal bundle construction of $\mathscr{P}_1\mathscr{C}$, in fact, the standard projective representation of states $\hat{\psi}_{\hat{x}}$ plays a mathematically

natural role, since it is equivalent to a choice of *connection* on the bundle space R_0^4. Each choice of $\hat{\psi}_{\hat{x}}$ is a projection onto the fiber $\hat{x}$ (*vertical space* of R_0^4) and so its complementary projection $\psi_{\hat{x}}^{\perp} = I - \hat{\psi}_{\hat{x}}$ defines the corresponding *horizontal subspace* of R_0^4. Spreading such choices out along a fibre defines a splitting of the tangent space $R_x^4 \sim R^4$ at $x \in R_0^4$ into a vertical component V_x^{ψ} (range of $\hat{\psi}_x$) and a horizontal component $H_x^{\psi} = V_x^{\psi^{\perp}}$ (range of $\hat{\psi}_x^{\perp}$) so that $R_x^4 = R^4 = V_x^{\psi} \oplus H_x^{\psi}$, and when such choices are made smoothly this is a standard construction of a connection.

The choice of a connection now allows the definition of *horizontal* (*covariant*) and *vertical* derivatives: for any mapping f defined on R_0^4 and any tangent vector field $y_x \in T_x R_0^4 = R_x^4$,

$$D_x^{\psi^{\perp}} f \cdot y_x \equiv D_x f \cdot \psi_x^{\perp} y_x \equiv D_x^H f \cdot y_x$$

$$D_x^{\psi} f \cdot y_x \equiv D_x f \cdot \psi_x y_x \equiv D_x^V f \cdot y_x$$

To work with the corresponding Lie-algebra-valued one-form definition of connection, we simply make use of the natural action L_x of bundle points $x \in R_0^4$ on the gauge group $\mathscr{G}_0$:

$$L_x \colon \mathscr{G}_0 \to \hat{\pi}(x)$$

$$C \to L_x C = Cx$$

and its derivative:

$$D_x L \colon \mathscr{G} \to \hat{\pi}(x) \cup \{0\}$$

$$C \to D_x L \cdot C = Cx$$

Noting that L_x is invertible on the fiber $\hat{\pi}(x)$, we have

$$[D_x L]^{-1} \colon \hat{\pi}(x) \cup \{0\} \to \mathscr{G}$$

$$y \to [D_x L]^{-1} \cdot y = C \qquad \text{such that } y = Cx$$

Then $\omega_x = [D_x L]^{-1} \cdot \hat{\psi}_x$ defines the vertical $\mathscr{G}$-valued form on tangent vectors known as the *one-form of the connection*, and $d_x \omega \cdot \psi_x^{\perp} = d_x^H \omega$ defines the *curvature* of the connection.

It should now be evident that choosing a connection is equivalent to specifying a metric $g_x(\ ,\ )$ on the tangent bundle of R_0^4 (differentiably equivalent to the Euclidean metric and such that $i = -i^+$), since the

requirement $\hat{\psi}_x = \hat{\psi}_x^+$ defines such a metric, and this in turn is equivalent to specifying a "timelike" vector field $\gamma_0(x)$ in the Dirac algebra over R_0^4 so that $\gamma_0(x)\tilde{d}\gamma_0(x) = d^+$ defines g-adjointness on $\mathscr{D}$ relative to the Euclidean adjoint $\tilde{d}$. Application of the QR principle then requires invariance under all such metric (i.e., connection) choices, since they determine differentiably equivalent metrics on $\mathscr{P}_1\mathscr{C}$ via charts (x, X) on R_0^4 and horizontal lifts of tangent vectors at $\hat{x}$ on $\mathscr{P}_1\mathscr{C}$ to tangent vectors at x over R_0^4. As Bleecker (1981) shows in full generality, if standard Lagrangian variational methods are used over R_0^4 to guarantee the required invariance (by incorporating the scalar curvature of the bundle metric connection into Lagrangians in the usual way), then both the Einstein field equations for g and the Yang–Mills equation for ω hold over R_0^4, and geodesics x_t in the four-dimensional bundle manifold project down to generally nongeodesic quantum state trajectories $\hat{\pi}(x_t) = \hat{x}_t$ in the base manifold $\mathscr{P}_1\mathscr{C}$ with *constant* connection form values $\omega_{x_t} \cdot \dot{x}_t = m + ie \in \mathscr{C}$. Thus we have a Kaluza–Klein-type quantum field theoretical model over a purely quantum theoretical base space $\mathscr{P}_1\mathscr{C}$, with the intriguing difference that no artificial fifth-dimension need be introduced since everything occurs naturally in the four-dimensional canonical bundle structure over $\mathscr{P}_1\mathscr{C}$, and we have two gauge Lie algebra "charge" parameters to interpret, instead of the usual single imaginary component e of the standard Kaluza–Klein model.

Since gauge quantum field theory supplies a well-established interpretation for the $\mathscr{U}(1) = \{e^{i\alpha(x)}\}$ component of the gauge group $\mathscr{C}_0$ acting on $\mathscr{P}_1\mathscr{C}$ systems—Weyl's gauge theory of the electromagnetic potential for the spin-1/2 electronic structure of matter incorporated into the relativistic Dirac spinor representation of such structure—the QR principle of economy requires us to use the same interpretation for field theory over $\mathscr{P}_1\mathscr{C}$, with of course the result that the pure imaginary component of the connection form ω on geodesics of R_0^4 is to be interpreted as electric charge. Can a similar case be made for the interpretation of the real component of ω as mass (the geodesically constant Lie algebra component of ω_x corresponding to the noncompact $e^{\rho(x)}$ part of the gauge group $\mathscr{C}_0$)? We believe the answer should be affirmative, given the already existing interpretation of spinorial quantities in both quantum theory and general relativity to represent the electromagnetic interactions of spin-1/2 matter. The simplest way to incorporate this standard interpretation is to identify the natural charts

$$\left(\theta(\hat{\pi}(x)), \varphi(\hat{\pi}(x)), e^{\rho(x)}, e^{i\alpha(x)}\right)$$

constructed earlier for the bundle space R_0^4 with the spinor representation (Penrose flagpole–flag picture) of pure frequency electromagnetic field emission and absorption as presented, for example, in Misner, Thorne, and Wheeler, *Gravitation* (1973), p. 1157ff., to describe laser ranging phenomena. In that picture, θ and φ represent, as before, standard lab polar angle

coordinates for the polarization vector associated with a $\mathscr{P}_1\mathscr{C}$ state (i.e., spin-1/2 structure), but interpreted in the canonical line bundle representing the simplest vector fields associated with such structures (electromagnetic fields) as specifying the lightlike (i.e., $r = t$) direction of propagation of the fields between the system states $\hat{x} \in \mathscr{P}_1\mathscr{C}$ (represented by zero section of line bundle) and possible observer points x (represented by nonzero sections necessary in the construction of bundle charts). This natural identification then forces the interpretation of $r(x) = t(x) = e^{\rho(x)}$ as the positive radial separation of system and observer in the four-dimensional bundle manifold R_0^4, and gives of course the already noted identification of $e^{i\alpha(x)}$ with the electromagnetic "flag on the flagpole" defined by (r, θ, φ), and we get as an additional bonus the fact that gauge changes in the positive operator $e^{\rho(x)}I \in GL(2, \mathscr{C})$ are associated in just the right way with general relativistic metric variations over R_0^4 (and $\mathscr{P}_1\mathscr{C}$) that *cannot* be defined by the special relativistic transformations of $SL(2, \mathscr{C})$. In this way, the geodesically constant vertical Lie algebra valued forms on the bundle space R_0^4 define natural meanings for both mass and charge over $\mathscr{P}_1\mathscr{C}$.

All these field theoretical arguments need to be spelled out in full detail in terms of general relativistic Lagrangians over R_0^4, of course, and we propose to present such a model in the near future, but for the present we can see that the "microcosmic" geometry of a simple $\mathscr{P}_1\mathscr{C}$ quantum system exhibits the full richness of general relativistic mathematical structure usually associated only with "macrocosmic" systems, and to the extent that we can understand our macroscopic thermodynamic worldview as built up from countless prismatic reflections of the fundamental $\mathscr{P}_1\mathscr{C}$ geometry we have a natural quantum origin for our general relativistic 4-space viewpoint —we live in general relativistic 4-space because they (fundamental observed systems) live in $\mathscr{P}_1\mathscr{C}$. On the other hand we can also see how such a worldview must prove inadequate when we encounter quantum systems (whether neutron or neutron star) that essentially inhabit $\mathscr{P}_2\mathscr{C}$, with the corresponding symmetry group $GL(3, \mathscr{C})$. This group contains $\mathscr{C}_0$, $SU(3)$, and $SL(3, \mathscr{C})$–$SU(3)$ uniquely and naturally, but any appearance of $GL(2, \mathscr{C}) = \mathscr{C}_0 \times SL(2, \mathscr{C})$ will be nonunique and arbitrary, and the attempt to represent the essentially six-dimensional geometry of $\mathscr{P}_2\mathscr{C}$ systems in terms of the four-dimensional general relativistic geometry appropriate to $\mathscr{P}_1\mathscr{C}$ will obviously result in mathematical singularities.

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